Math 247A Lecture 24 Notes

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1 Estimating Oscillatory Integrals With Stationary Phase

1.1 Estimation in the 1 dimensional case

Proposition 1.1 (stationary phase)**.** *Assume $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and has a non-degenerate critical points at x_0; that is, $\phi'(x_0) = 0$ and $\phi''(x_0) \neq 0$. Assume $\psi : \mathbb{R} \to \mathbb{C}$ is smooth and supported in a sufficiently small neighborhood of x_0. Then*

$$\begin{aligned} I(\lambda) &= \int e^{i\lambda\phi(x)}\psi(x)\,dx \\ &= e^{i\lambda\phi(x_0)}\psi(x_0)\sqrt{2\pi}e^{i(\pi/4)\operatorname{sgn}(\phi''(x_0))}|\phi''(x_0)|^{-1/2}|^{-1/2}\lambda^{-1/2} + O(\lambda^{-3/2}) \end{aligned}$$

as $\lambda \to \infty$.

Remark 1.1. If we are not interested in the coefficient of the leading order term, then we can argue as follows: Let $a \in C_c^\infty$ be such that

$$a(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 2 \end{cases}$$

and decompose

$$I(\lambda) = I_1(\lambda) + I_2(\lambda),$$

$$I_1(\lambda) = \int e^{i\lambda\phi(x)}\psi(a)a(\lambda^{1/2}(x - x_0))\,dx.$$

Then

$$\begin{aligned} |I_1(\lambda)| &\leq \|\psi\|_\infty \int |a(\lambda^{1/2}(x - x_0))|\,dx \\ &\leq \|\psi\|_\infty\|a\|_\infty \cdot \lambda^{1/2}, \end{aligned}$$

$$I_2(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)[1 - a(\lambda^{1/2}(x - x_0))]\,dx.$$

Note that $\operatorname{supp}(\psi(x)[1-a(\lambda^{1/2}(x-x_0))]) \subseteq \{\lambda^{-1/2} \le |x-x_0| \lesssim_\psi 1\}$. If $\operatorname{supp}\psi$ is such that $\phi'(x) \neq 0$ for $x \in (\operatorname{supp}\psi) \setminus \{x_0\}$, then integration by parts gives

$$|I_2(\lambda)| \lesssim_m \lambda^{-m} \qquad \forall m \ge 0.$$

Proof. Write

$$\phi(x) = \phi(x_0) + \cancelto{0}{\phi'(x_0)(x-x_0)} + \frac{\phi''(x_0)}{2}(x-x_0)^2 + O(|x-x_0|^3).$$

Rewrite this as

$$\phi(x) - \phi(x_0) = \frac{\phi''(x_0)}{2}(x-x_0)^2[1+\eta(x)],$$

where $\eta(x) = O(|x-x_0|)$. Let U be a small neighborhood of x_0 such that

1. $|\eta(x)| < 1$ for all $x \in U$
2. $\phi'(x) \neq 0$ for all $x \in U \setminus \{x_0\}$.

Assume $\operatorname{supp}\psi \subseteq U$. Change variables to $y(x) = (x-x_0)\sqrt{1+\eta(x)}$. This is a diffeomorphism from U to a neighborhood of $y=0$. Then

$$\begin{aligned} I(\lambda) &= e^{i\lambda\phi(x_0)} \int e^{i\lambda[\phi(x)-\phi(x_0)]}\psi(x)\,dx \\ &= e^{i\lambda\phi(x_0)} \int e^{i\lambda(\phi''(x_0)/2)y^2}\widetilde{\psi}(y)\,dy, \end{aligned}$$

where $\widetilde{\psi} \in C_c^\infty$ is supported in a neighborhood of $y=0$ and $\widetilde{\psi}(0) = \psi(x_0)$.

Let $\widetilde{\widetilde{\psi}} \in C_c^\infty$ be such that $\widetilde{\widetilde{\psi}} = 1$ on $\operatorname{supp}\widetilde{\psi}$. Let $\widetilde{\lambda} = \frac{\lambda\phi''(x_0)}{2}$. Then

$$I(\lambda) = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} e^{-y^2}[e^{y^2}\widetilde{\psi}(y)]\widetilde{\widetilde{\psi}}(y)\,dy$$

Using a Taylor expansion, we write

$$e^{y^2}\widetilde{\psi}(y) = \sum_{j=0}^N a_j y^j + y^{N+1}R_N(y), \qquad R_N(y) = \frac{1}{N!}\int_0^1 (1-t)^N \frac{d^{N+1}}{dy^{N+1}}[e^{|\cdot|^2}\widetilde{\psi}](ty)\,dt.$$

This leads us to consider 3 terms:

$$\mathrm{I} = e^{i\lambda\phi(x_0)} \sum_{j=0}^N a_j \int e^{i\widetilde{\lambda}y^2} e^{-y^2} y^j\,dy,$$

$$\mathrm{II} = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} e^{-y^2} P_N(y)[\widetilde{\widetilde{\psi}}(y) - 1]\,dy,$$

$$\mathrm{III} = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} e^{-y^2} y^{N+1} R_N 9y) \widetilde{\widetilde{\psi}}(y)\, dy.$$

Since $\frac{d}{dy} e^{i\widetilde{\lambda}y^2}$, we can pull off a factor of y using integration by parts. By picking N to be large enough, we can get as much decay in III as we want.

Let's look at I. Note that the terms with j odd vanish. Consider $j = 0$ and note that $a_0 = \psi(x_0)$. The contribution is

$$e^{i\lambda\phi(x_0)} \psi(x_0) \int e^{i\widetilde{\lambda}y^2} e^{-y^2}\, dy = e^{i\lambda\phi(x_0)} \psi(x_0) (1 - i\widetilde{\lambda})^{1/2} \sqrt{\pi}.$$

To see what happens when $\lambda \to \infty$, write $1 - i\widetilde{\lambda} = re^{i\sigma}$, where $r = \sqrt{1 + \widetilde{\lambda}^2}$ and $\tan\sigma = -\widetilde{\lambda}$. Then $(1 - i\widetilde{\lambda})^{-1/2} = r^{-1/2} e^{-i\sigma/2}$. Then

$$\begin{aligned} r^{-1/2} &= \left(|\widetilde{\lambda}| \sqrt{1 + \frac{1}{\widetilde{\lambda}^2}} \right)^{-1/2} \\ &= |\widetilde{\lambda}|^{-1/2} \left(1 + \frac{1}{\widetilde{\lambda}^2} \right)^{-1/4} \\ &= \left(\frac{\lambda|\phi''(x_0)|}{2} \right)^{-1/2} \left(1 + O(\lambda^{-2})\right) \\ &= \left(\frac{2}{\lambda|\phi''(x_0)|} \right)^{1/2} + O(\lambda^{-5/2}), \end{aligned}$$

$$\tan\sigma = -\widetilde{\lambda} = -\frac{\lambda\phi''(x_0)}{2} \xrightarrow{\lambda\to\infty} -\operatorname{sgn}(\phi''(x_0)) \cdot \infty.$$

So $\sigma \to -\operatorname{sgn}(\phi''(x_0)) \cdot \frac{\pi}{2}$, and we get

$$e^{i\lambda\phi(x_0)} \psi(x_0) \sqrt{\frac{2\pi}{\lambda|\phi''(x_0)|}} e^{\operatorname{sgn}(\phi''(x_0))\pi/4} + O(\lambda^{-5/2}).$$

For $j \geq 2$ even,

$$\left| \int e^{i\widetilde{\lambda}y^2} e^{-y^2} y^j\, dy \right| = \left| (1 - i\widetilde{\lambda})^{-1/2 - j/2} \int e^{-y^2} y^j\, dy \right| \lesssim \lambda^{-(j+1)/2} \lesssim \lambda^{-3/2}.$$

Now consider II. Note that $y \mapsto e^{-y^2} P_N(y)[\widetilde{\widetilde{\psi}}(y) - 1]$ is supported away from the origin. Integration by parts gives

$$|\,\mathrm{II}\,| \lesssim_m \lambda^{-m} \qquad \forall m \geq 0.$$

Consider III. Decompose $\mathrm{III} = \mathrm{III}_1 + \mathrm{III}_2$, where

$$\mathrm{III}_1 = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} e^{-y^2} y^{N+1} R_N(y) \widetilde{\widetilde{\psi}}(y) a(y/\varepsilon)\, dy.$$

Then

$$|\,\mathrm{III}_1\,| \lesssim \int_{|y|\leq\varepsilon} |y|^{N+1}\,dy \lesssim \varepsilon^{N+2}.$$

The other term is

$$\mathrm{III}_2 = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} y^{N+1}[1 - a(y/\varepsilon]b(y)\,dy,$$

where $b(y) = e^{-y^2}R_N(y)\widetilde{\widetilde{\psi}}(y)$. Integration by parts gives

$$\mathrm{III}_2 = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} \cdot \left(-\frac{d}{dy}\frac{1}{2i\widetilde{\lambda}y}\right)^m [y^{N+1}(1 - ay/\varepsilon b(y)]\,dy,$$

so

$$\begin{aligned}
|\,\mathrm{III}_2\,| &\lesssim_m \frac{1}{\lambda^m}\sum_{k=0}^{m}\sum_{\alpha_1+\alpha_2+\alpha_3=m-k}\left\|\frac{y^{N+1-\alpha_1}\varepsilon^{-\alpha_2}a^{(\alpha_2)}(y/\varepsilon)b^{(\alpha_3)}(y)}{y^{m+k}}\right\|_{L^1} \\
&\lesssim_m \frac{1}{\lambda^m}\int_{|y|\geq\varepsilon} |y|^{N+1-2m}\,dy \\
&\lesssim \frac{\varepsilon^{N+2-2m}}{\lambda^m}
\end{aligned}$$

if $m > \frac{N+2}{2}$. Now choose ε such that

$$\varepsilon^{N+2} = \frac{\varepsilon^{N+2-2m}}{\lambda^m} \iff \varepsilon = \lambda^{-1/2}$$

to get $|\,\mathrm{III}\,| \lesssim \lambda^{-(N+2)/2} \lesssim \lambda^{-3/2}$ if $N \geq 1$. $\square$